

Invariant cocalibrated \mathbf{G}_2 -structures on nilmanifolds

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Abstract

We classify nilmanifolds admitting invariant cocalibrated \mathbf{G}_2 -structures.

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1 Introduction

A seven dimensional connected, oriented, Riemannian manifold M with Holonomy contained in \mathbf{G}_2 is characterized by existence of a \mathbf{G}_2 -reduction \mathcal{P} of its orthogonal frame bundle \mathcal{F} , on which the Cartan form of the Levi-Civita connection restricts; in other words by the vanishing of the intrinsic torsion τ of \mathcal{P} , [21].

Alternatively one can observe that $\mathbf{G}_2 \subset \mathbf{SO}(7)$ is defined as the stabiliser, under the $\mathbf{GL}(7, \mathbb{R})$ -action, of a stable 3-form φ_0 , or equivalently (the component of) the stabiliser of a stable 4-form ϕ_0 , both defining the underlying euclidean metric and related by $\phi_0 = *\varphi_0$.

Consequently the principal bundle \mathcal{P} is defined by a form φ , or ϕ , satisfying $\xi_*\varphi = \varphi_0$ and $\xi_*\phi = \phi_0$ for a suitable frame ξ . Then τ is related to $\nabla\varphi$, or $\nabla\phi$, and its vanishing is equivalent to both $d\varphi = 0$ and $d\phi = 0$, [11].

For general linear principal \mathbf{G}_2 -bundles, τ equals $\tau_0 + \tau_1 + \tau_2 + \tau_3$, where each τ_h takes values in a bundle modelled on an irreducible \mathbf{G}_2 -module χ_h , [11]. Conditions $d\varphi = 0$ and $d\phi = 0$ are equivalent to $\tau_0 = \tau_1 = \tau_3 = 0$ and $\tau_1 = \tau_2 = 0$ respectively; if the first

holds we say the structure *calibrated* and, if the second holds, *cocalibrated*. For instance codimension one submanifolds of **Spin**(7)-manifolds and closed seven dimensional spin manifolds have naturally cocalibrated \mathbf{G}_2 -structures [7].

It is well known that other combinations of torsions are admissible, see [9], [2], [3], [8].

Generally the calibrated condition is more restrictive than its counterpart, in fact compact examples of manifolds with calibrated structures are not easy to exhibit. In [5] Conti and Fernández classified nilmanifolds with a (left) invariant calibrated \mathbf{G}_2 -structure. Some homogeneous cocalibrated structure was studied in [2], and by Freibert in [13] and [12]. In this paper we present the classification of nilmanifolds admitting cocalibrated structures. Any nilmanifold equipped with an invariant metric is spin, thus, by Theorem 1.8 from [7], it has a cocalibrated \mathbf{G}_2 -structure. As a result we obtain that nilmanifolds defined by Lie algebras in Tables 2, 3, 8 and 9 admit a cocalibrated \mathbf{G}_2 -structure, but not an invariant one.

A nilmanifold is a compact manifold on which a nilpotent Lie group G acts transitively, namely homogeneous of type G/N with N a discrete subgroup [18]. The Lie group G can define a nilmanifold if and only if its Lie algebra \mathfrak{g} admits a basis $\{e_1, \dots, e_7\}$ with rational structure constants; in this case the De Rham cohomology of G/N can be calculate using invariant differential forms and it is isomorphic to the Chevalley-Eilenberg cohomology of \mathfrak{g} , see [19]. Invariant forms on G/N are uniquely determined by forms on \mathfrak{g} , hence we can restrict our attention to seven dimensional nilpotent Lie algebras (classified in [14]).

The paper is structured as follows.

In Section § 2 we recall theory of stable forms in order to describe algebraic properties of \mathbf{G}_2 .

In Section § 3 we classify \mathbf{G}_2 -structures on seven dimensional nilpotent Lie algebras with respect to particular fibrations over six dimensional nilpotent Lie algebras equipped with **SU**(3)-structures, and produce some obstructions to their existence.

Finally in Sections § 4 and § 5 we classify all seven dimensional nilpotent Lie algebras admitting cocalibrated \mathbf{G}_2 -structures.

Appendix § 7 contains some explicit calculations omitted from the rest of the paper.

2 Stable forms

In this section we describe some algebraic properties of \mathbf{G}_2 in terms of stable forms. We start recalling few classical general results on \mathbf{GL} -orbits of forms ([17]) and next we focus on form spaces over six and seven dimensional real vector spaces ([20]). For a more detailed treatment see [6].

Consider the natural $\mathbf{GL}(V)$ -action on $\Lambda^k V^*$.

A k -form ρ on an n -dimensional real vector space V is said to be *stable* if its orbit is open in $\Lambda^k V^*$.

Theorem 2.1. *Suppose $k \leq [\frac{n}{2}]$. Then stable k -forms (equivalently $(n-k)$ -forms) exist if and only if $k \leq 2$ or $k = 3$ and $n \in \{6, 7, 8\}$.*

Moreover, k fixed, there are finite many open orbits.

Theorem 2.2. *Suppose n even and $k \in \{2, n-2\}$ or $n \in \{6, 7, 8\}$ and $k \in \{3, n-3\}$. Then there exists an $\mathbf{GL}(V)$ -equivariant map*

$$\varepsilon : \Lambda^k V^* \rightarrow \Lambda^n V^*, \quad (2.1)$$

homogeneous of degree $(\frac{n}{k})$, such that $\text{Ker}(\varepsilon) = \{\text{non stable forms}\}$.

Moreover if we consider a stable k -form ρ and define the dual form $\hat{\rho} \in \Lambda^{n-k} V^$ as*

$$d\varepsilon_\rho(\alpha) = \hat{\rho} \wedge \alpha, \quad \forall \alpha \in \Lambda^k V^*,$$

then $\hat{\rho}$ is stable, the identity component of its stabiliser equals that one of ρ and verifies

$$\hat{\rho} \wedge \rho = \frac{n}{k} \varepsilon(\rho).$$

2.1 Stable forms in six dimensions

Consider $n = 6$.

In $\Lambda^2 V^*$ there exists only one open orbit. In particular we have the following

Theorem 2.3. *The set*

$$\Lambda_0(V) = \{\omega \in \Lambda^2 V^* \mid \omega^3 \neq 0\},$$

is the only open orbit in $\Lambda^2 V^$. Thus, given $\omega \in \Lambda_0(V)$, its stabiliser is isomorphic to $\mathbf{Sp}(6, \mathbb{R})$, its volume form $\varepsilon(\omega)$ can be chosen equal to $\frac{1}{3!}\omega^3$ and its dual form σ equal to $\frac{1}{2}\omega^2$.*

Moreover, there exists a suitable co-frame $\{f^1, \dots, f^6\}$ such that

$$\begin{aligned}\omega &= f^{12} + f^{34} + f^{56}, \\ \sigma &= f^{1234} + f^{1256} + f^{3456}, \\ \varepsilon(\omega) &= f^{123456}.\end{aligned}$$

The space $\Lambda^3 V^*$ has, instead, two different open orbits. They can be distinguished as follows (see [16]). Consider $\rho \in \Lambda^3 V^*$ and define¹

$$k_\rho : V \ni x \mapsto \iota_x \rho \wedge \rho \in \Lambda^5 V^*.$$

Thanks to the isomorphism $\theta^{-1} : V \otimes \Lambda^6 V^* \ni x \otimes \alpha \mapsto \iota_x \alpha \in \Lambda^5 V^*$ we can define

$$\begin{aligned}K_\rho &: V \xrightarrow{k_\rho} \Lambda^5 V^* \xrightarrow{\theta} V \otimes \Lambda^6 V^*, \\ &\text{and} \\ \lambda(\rho) &= \frac{1}{6} \text{trace}(K_\rho^2) \in (\Lambda^6 V^*)^{\otimes 2}.\end{aligned}$$

Theorem 2.4. *Consider the sets²*

$$\Lambda_+(V) = \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) > 0\} \quad \text{and} \quad \Lambda_-(V) = \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) < 0\}.$$

They are the only open orbits in $\Lambda^3 V^$.*

If $\psi \in \Lambda_+(V)$ the identity component of its stabiliser is isomorphic to $\mathbf{SL}(3, \mathbb{R}) \times \mathbf{SL}(3, \mathbb{R})$ and in a suitable co-frame $\{f^1, \dots, f^6\}$ results

$$\psi = f^{123} + f^{456}.$$

If $\psi \in \Lambda_-(V)$ the identity component of its stabiliser is isomorphic to $\mathbf{SL}(3, \mathbb{C})$ and in a suitable co-frame $\{f^1, \dots, f^6\}$ results

$$\psi = -f^{246} + f^{136} + f^{145} + f^{235}.$$

Moreover, if we consider a stable 3-form ψ , fix a volume form $\varepsilon(\psi) \in \Lambda^6 V^$ and define*

$$J_\psi = \frac{1}{\sqrt{|\lambda(\psi)|}} K_\psi \in V^* \otimes V,$$

then

$$\begin{aligned}\psi \in \Lambda_+(V) &\Leftrightarrow J_\psi \text{ is a para-complex structure,} \\ \psi \in \Lambda_-(V) &\Leftrightarrow J_\psi \text{ is a complex structure,}\end{aligned}$$

and in both cases the dual form is $J_\psi^ \psi$.*

¹Let ι_x be the contraction by the vector x .

²We say $\alpha \in (\Lambda^6 V^*)^{\otimes 2}$ positive (resp. negative) and write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha = \varepsilon \otimes \varepsilon$ (resp. $\alpha = -\varepsilon \otimes \varepsilon$).

For our purpose we are interested in particular pairs of stable forms, described by the following

Theorem 2.5. *Let $(\omega, \psi_-) \in \Lambda_0(V) \times \Lambda_-(V)$, J_{ψ_-} defined by the choose of $\varepsilon(\omega) \in \Lambda^6 V^*$, $h \in V^* \otimes V^*$ defined by*

$$h(x, y) = \omega(x, J_{\psi_-} y), \quad \forall x, y \in V,$$

and suppose

$$\omega \wedge \psi_- = 0.$$

If h is positive definite then the stabiliser of the pair (ω, ψ_-) is a subgroup of $\mathbf{SO}(V, h)$ isomorphic to $\mathbf{SU}(3)$, i.e. the pair defines an $\mathbf{SU}(3)$ -structure where J_{ψ_-} is the complex structure, $\Psi = -J_{\psi_-}^ \psi_- + i\psi_-$ the complex volume form and h the underlying hermitian metric.*

Further any other $\mathbf{SU}(3)$ -structure is obtained in this way.

Moreover, considering the real part ψ_+ of Ψ , if (ω, ψ_-) is normalized,

$$\text{i.e. } \psi_+ \wedge \psi_- = \frac{2}{3} \omega^3,$$

then

$$\sigma = *_h \omega,$$

$$\psi_+ = *_h \psi_-,$$

and there exists a suitable (h -orthonormal) co-frame $\{f^1, \dots, f^6\}$ such that

$$\begin{aligned} \omega &= f^{12} + f^{34} + f^{56}, \\ \psi_- &= -f^{246} + f^{136} + f^{145} + f^{235}. \end{aligned} \tag{2.2}$$

2.2 Stable forms in seven dimensions

Consider $n = 7$.

Given a 3-form φ consider the symmetric 2-form b_φ on V with values in $\Lambda^7 V^*$,

$$b_\varphi(x, y) = \iota_x \varphi \wedge \iota_y \varphi \wedge \varphi, \quad \forall x, y \in V.$$

Then the volume form map (2.1) can be defined as

$$\varepsilon(\varphi) = \sqrt[9]{\det(b_\varphi)}.$$

Theorem 2.6. *Let φ be any stable 3-form and g_φ the symmetric 2-form on V defined by*

$$g_\varphi = \frac{1}{3\varepsilon(\varphi)} b_\varphi.$$

Then

$$\Pi_+(V) = \{\varphi \in \Lambda^3 V^* \mid g_\varphi \text{ is definite}\} \text{ and } \Pi_-(V) = \{\varphi \in \Lambda^3 V^* \mid g_\varphi \text{ is indefinite}\},$$

are the only open orbits in $\Lambda^3 V^$.*

*If $\varphi \in \Pi_+(V)$ we say it (and its dual form) positive, its stabiliser is a subgroup of $\mathbf{SO}(V, g_\varphi)$ isomorphic to \mathbf{G}_2 , its dual form ϕ equals $*_{g_\varphi} \varphi$ and in a suitable (g_φ -orthonormal) co-frame $\{f^1, \dots, f^7\}$ results*

$$\begin{aligned} \varphi &= f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}, \\ \phi &= f^{1234} + f^{1256} + f^{3456} + f^{2467} - f^{1367} - f^{1457} - f^{2357}. \end{aligned} \tag{2.3}$$

Remark 1. Since the stabiliser of a positive 4-form is isomorphic to $\mathbf{G}_2 \times \mathbb{Z}_2$, and the correspondence between positive forms $\{\varphi \mapsto \phi\}$ is $2 : 1$, a \mathbf{G}_2 -structure is defined by ϕ together with an orientation (see [1]).

3 Cocalibrated structures

In this section we classify seven dimensional nilpotent Lie algebras with a cocalibrated \mathbf{G}_2 -structure in terms of fibrations over six dimensional nilpotent Lie algebras endowed with a particular $\mathbf{SU}(3)$ -structure.

Consider an oriented seven dimensional nilpotent real Lie algebra \mathfrak{g} with center \mathfrak{z} and volume form $\varepsilon \in \Lambda^7 \mathfrak{g}^*$, a non zero vector $X \in \mathfrak{z}$ and the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R}X \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0. \quad (3.1)$$

Remark 2. There is a natural isomorphism of real algebras

$$\{\alpha \in \Lambda^\bullet \mathfrak{g}^* \mid \iota_X \alpha = 0\} \xrightarrow{\pi_*} \Lambda^\bullet \mathfrak{h}^*.$$

Proposition 3.1. *Let ϕ be a stable 4-form on \mathfrak{g} defining a cocalibrated \mathbf{G}_2 -structure compatible with the orientation, g the underlying metric and η the dual form of $\frac{1}{\|X\|}X$ with respect to g .*

Then the pair of forms $(\omega, \psi_-) \in \Lambda^2 \mathfrak{h}^ \times \Lambda^3 \mathfrak{h}^*$,*

$$\begin{aligned} \psi_- &= \pi_* \left(-\frac{1}{\|X\|} \iota_X \phi \right), \\ \omega \text{ s.t. } &\begin{cases} \frac{1}{6} \omega^3 = \pi_* \left(\frac{1}{\|X\|} \iota_X \varepsilon \right), \\ \sigma = \frac{1}{2} \omega^2 = \pi_* \left(\phi + \frac{1}{\|X\|} \iota_X \phi \wedge \eta \right), \end{cases} \end{aligned}$$

defines an $\mathbf{SU}(3)$ -structure on \mathfrak{h} , is normalized, and satisfies³

$$\begin{aligned} d(\pi^* \psi_-) &= 0, \\ d(\pi^* \sigma) &= (\pi^* \psi_-) \wedge d(\eta), \end{aligned} \quad (3.2)$$

$$g = \pi^* h + \eta \cdot \eta, \quad (3.3)$$

$$\phi = \pi^* \sigma + (\pi^* \psi_-) \wedge \eta. \quad (3.4)$$

Vice versa let $(\omega, \psi_-) \in \Lambda^2 \mathfrak{h}^ \times \Lambda^3 \mathfrak{h}^*$ be a pair of stable forms defining an $\mathbf{SU}(3)$ -structure on \mathfrak{h} , compatible with the orientation $\pi_*(\iota_X \varepsilon)$ and normalized; $\eta \in \Lambda^1 \mathfrak{g}^*$ a 1-form for which Equation (3.2) is satisfied and the symmetric 2-form $\pi^* h + \eta \cdot \eta$ is positive definite. Then the 4-form ϕ defined by (3.4) induces a cocalibrated \mathbf{G}_2 -structure with underlying metric g defined by (3.3) and volume form ε .*

Proof. Define

$$\begin{aligned} \psi_- &= -\frac{1}{\|X\|} \iota_X \phi, \\ \sigma &= \phi - \psi_- \wedge \eta. \end{aligned}$$

It is easy to see⁴ that those are stable forms inducing an $\mathbf{SU}(3)$ -structure on X^\perp , and are normalized. Moreover, because of both ψ_- and σ vanish on X , \mathfrak{h} inherits an $\mathbf{SU}(3)$ -structure defined by the two stable forms $\pi_* \psi_-$ and $\pi_* \sigma$, which we will denote as ψ_- and σ , and such that π is an isometry on X^\perp .

From $L_X = 0$ we deduce that forms ψ_- and σ satisfy Equation (3.2), in fact

$$\begin{aligned} d\psi_- &= d \left(-\frac{1}{\|X\|} \iota_X \phi \right) = -\frac{1}{\|X\|} \iota_X (d(\phi)) = 0, \\ d\sigma &= d(\phi - \psi_- \wedge \eta) = \psi_- \wedge d(\eta), \end{aligned}$$

³Recall notation in § 2.1.

⁴Compare normal forms (2.3) and (2.2).

⁵Here X^\perp indicates the g -orthogonal complement of $\{X\}$ in \mathfrak{g} .

and the metric g satisfies Equation (3.3).

For the converse suppose ω, ψ_-, η as in the statement. By definition X^\perp admits an $\mathbf{SU}(3)$ -structure defined by the pair $(\pi^*\omega, \pi^*\psi_-)$. Consider a π^*h -orthonormal frame $\{f_1, \dots, f_6\}$ of X^\perp such that (2.2) holds and put $f_7 = \frac{1}{\|X\|}X$. Then the 4-form ϕ defined by Equation (3.4) is stable and induces the metric g , in fact it has the normal form (2.3) with respect to the co-frame $\{f^1, \dots, f^7\}$. Furthermore Equation (3.2) implies $d(\phi) = 0$. \square

Thus we have the following obstruction to existence of cocalibrated \mathbf{G}_2 -structures on nilpotent Lie algebras:

Corollary 3.2. *Let*

$$Z_X^3 = \{\pi_*(\iota_X \alpha) \in \Lambda^3 \mathfrak{h}^* \mid \alpha \in \Lambda^4 \mathfrak{g}^*, d(\alpha) = 0\}.$$

If

$$Z_X^3 \cap \Lambda_-(\mathfrak{h}) = \emptyset$$

then \mathfrak{g} does not admit a cocalibrated \mathbf{G}_2 -structure.

Remark 3. We will see that, except for the nilpotent Lie algebras listed in Tables 3 and 9, when \mathfrak{g} does not satisfy hypothesis of Corollary 3.2, then cocalibrated \mathbf{G}_2 -structures exist.

Another result which will be useful later is the following

Lemma 3.3. *Let (ω, J) be an $\mathbf{SU}(3)$ -structure on a six dimensional vector space V with fundamental 2-form ω , orthogonal complex structure J and hermitian metric h . Then for any J -invariant 4-dimensional subspace W of V results $\sigma|_W \neq 0$, where σ is the dual form of ω .*

Proof. Let $x, y \in W$ non zero vectors such that $\{x, Jx, y, Jy\}$ is a $h|_W$ -orthonormal real basis of the space W , and z a non zero unit vector in W^\perp . Then $\{x, Jx, y, Jy, z, Jz\}$ is a real h -orthonormal basis of V . It follows that

$$\sigma(x \wedge Jx \wedge y \wedge Jy) = (*_h \omega)(x \wedge Jx \wedge y \wedge Jy) = \omega(z \wedge Jz) = \|z\|^2 = 1,$$

thus $\sigma|_W \neq 0$. \square

4 Decomposable case

In this section we classify all seven dimensional decomposable nilpotent Lie algebras which admit a cocalibrated \mathbf{G}_2 -structure.

Let \mathfrak{g} be a seven dimensional decomposable Lie algebra and \mathfrak{z} its center. Recall that an $\mathbf{SU}(3)$ -structure with fundamental 2-form ω and complex volume form Ψ is said to be *Half-Flat* if and only if $d\omega = 0$ and $d\text{Re}(\Psi) = 0$.

Lemma 4.1. *Suppose \mathfrak{g} decomposes as $\mathfrak{h} \oplus \mathbb{R}$ where \mathfrak{h} is a six dimensional nilpotent Lie algebra admitting an Half-Flat $\mathbf{SU}(3)$ -structure and acting trivially on the factor \mathbb{R} by the adjoint representation. Then \mathfrak{g} admits a cocalibrated \mathbf{G}_2 -structure.*

Proof. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}e_7, \quad e_7 \in \mathfrak{z},$$

and $(\omega, \psi'_+ + i\psi'_-)$ an Half-Flat $\mathbf{SU}(3)$ -structure on \mathfrak{h} , where ω is the fundamental 2-form and ψ'_+, ψ'_- the real and imaginary part of the complex volume form. Observe that also $(\omega, -\psi'_- + i\psi'_+)$ defines an $\mathbf{SU}(3)$ -structure. Let η be the 1-form on \mathfrak{g} defined by

$$\begin{aligned} \eta(e_7) &= 1, \\ \eta(Y) &= 0, \quad \forall Y \in \mathfrak{h}. \end{aligned}$$

By hypothesis it verifies $d\eta = 0$. Therefore, putting

$$\begin{aligned}\omega &= \omega', \\ \psi_- &= \psi'_+, \end{aligned}$$

the triple (ω, ψ_-, η) satisfies conditions of Theorem 3.1, hence defines a cocalibrated \mathbf{G}_2 -structure. \square

By the notation

$$\mathfrak{g} = (de^1, \dots, de^7) = \left(\sum_{i_1 j_1} c_{i_1 j_1}^1 i_1 j_1, \dots, \sum_{i_7 j_7} c_{i_7 j_7}^7 i_7 j_7 \right),$$

we mean that $\{e^k\}$ is a basis of \mathfrak{g}^* such that $de^k = \sum_{i < j} c_{ij}^k e^{ij}$, where $c_{ij}^k \in \mathbb{R}$ and $e^{ij} = e^i \wedge e^j$.

Proposition 4.2. *Among all seven dimensional, decomposable, nilpotent, Lie algebras those admitting a cocalibrated \mathbf{G}_2 -structure arise from the construction described in Lemma 4.1. Explicitly*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R},$$

where \mathfrak{h} is a six dimensional, nilpotent, Lie algebra lying in Table 1.

Table 1: Six dimensional nilpotent Lie algebras admitting Half-Flat $\mathbf{SU}(3)$ -structures.

\mathfrak{h}
$(0, 0, 0, 0, 0, 0)$
2-step nilpotent Lie algebras
$(0, 0, 0, 12, 13, 23)$
$(0, 0, 0, 0, 13 - 24, 14 + 23)$
$(0, 0, 0, 0, 12, 14 + 23)$
$(0, 0, 0, 0, 12, 13)$
$(0, 0, 0, 0, 12, 34)$
$(0, 0, 0, 0, 0, 12 + 34)$
$(0, 0, 0, 0, 0, 12)$
3-step nilpotent Lie algebras
$(0, 0, 0, 0, 12, 15 + 34)$
$(0, 0, 0, 0, 12, 14 + 25)$
$(0, 0, 0, 12, 23, 14 + 35)$
$(0, 0, 0, 12, 23, 14 - 35)$
$(0, 0, 0, 12, 13, 14 + 35)$
$(0, 0, 0, 12, 13, 14 + 23)$
$(0, 0, 0, 12, 13, 24)$
4-step nilpotent Lie algebras
$(0, 0, 12, 13, 23, 14)$
$(0, 0, 12, 13, 23, 14 + 25)$
$(0, 0, 12, 13, 23, 14 - 25)$
$(0, 0, 0, 12, 14, 15 + 23)$
$(0, 0, 0, 12, 14 - 23, 15 + 34)$
$(0, 0, 0, 12, 14, 15)$
$(0, 0, 0, 12, 14, 15 + 24)$
$(0, 0, 0, 12, 14, 15 + 23 + 24)$
5-step nilpotent Lie algebras
$(0, 0, 12, 13, 14 + 23, 24 + 15)$

Proof. Let \mathfrak{g} be any decomposable seven dimensional nilpotent real Lie algebra.

If \mathfrak{g} lies in Table 2 then there exists a non zero vector $X \in \mathfrak{z}$ such that

$$Z_X^3 \cap \Lambda_-(\mathfrak{h}) = \emptyset,$$

hence, by Corollary 3.2, it does not admit a cocalibrated \mathbf{G}_2 -structure.

Table 2: Decomposable nilpotent Lie algebras satisfying $Z_X^3 \cap \Lambda_-(\mathfrak{h}) = \emptyset$.

\mathfrak{g}	X
3-step nilpotent Lie algebras	
$(0, 0, 0, 0, 23, 34, 36)$	e_7
$(0, 0, 0, 0, 12, 15, 0)$	e_6
$(0, 0, 0, 12, 13, 14, 0)$	e_5
$(0, 0, 0, 12, 14, 24, 0)$	e_7
5-step nilpotent Lie algebras	
$(0, 0, 12, 13, 14, 23 + 15, 0)$	e_6
$(0, 0, 12, 13, 14, 15, 0)$	e_6
$(0, 0, 12, 13, 14, 34 - 25, 0)$	e_7
$(0, 0, 12, 13, 14 + 23, 34 - 25, 0)$	e_7

If \mathfrak{g} appears in Table 3, let ϕ be any closed 4-form. Put $X = e_7$, η any 1-form satisfying $\eta(X) > 0$, and define, as in Proposition 3.1, forms σ and ψ_- on \mathfrak{h} .

By way of contradiction suppose ϕ stable. Then there exists a solution of Equations (3.2) and (3.3). Now, for any volume form on \mathfrak{h} , the equation

$$d(\psi_-) = 0,$$

forces the subspace $W = \text{Span}(\pi(e_3), \pi(e_4), \pi(e_5), \pi(e_6)) \subset \mathfrak{h}$ to be J_{ψ_-} -invariant. But one can check⁶ that the equation

$$d(\sigma) = \psi_- \wedge d(\eta)$$

implies $\sigma|_W = 0$, contradicting Lemma 3.3.

Table 3: Decomposable nilpotent Lie algebras satisfying $Z_X^3 \cap \Lambda_-(\mathfrak{h}) \neq \emptyset$ for all $X \in \mathfrak{z}$ but admitting no cocalibrated \mathbf{G}_2 -structures.

\mathfrak{g}
3-step nilpotent Lie algebras
$(0, 0, 0, 12, 13 - 24, 14 + 23, 0)$
$(0, 0, 0, 12, 14, 13 - 24, 0)$
$(0, 0, 0, 12, 13 + 14, 24, 0)$

Any other \mathfrak{g} is the central extension of a six dimensional nilpotent Lie algebra \mathfrak{h} appearing in Table 1. Such \mathfrak{h} has an Half-Flat $\mathbf{SU}(3)$ -structure (see [4]), hence, by Lemma 4.1, \mathfrak{g} admits a cocalibrated \mathbf{G}_2 -structure. \square

⁶Explicit calculations can be found in § 7.

5 Indecomposable case

In this section we classify all seven dimensional indecomposable nilpotent Lie algebras which admit a cocalibrated \mathbf{G}_2 -structure.

Let \mathfrak{g} be a seven dimensional decomposable Lie algebra and \mathfrak{z} its center.

Proposition 5.1. *Among all seven dimensional, indecomposable, nilpotent, Lie algebras those admitting a cocalibrated \mathbf{G}_2 -structure are listed in tables 4 - 7.*

Table 4: Indecomposable 2-step nilpotent Lie algebras admitting cocalibrated \mathbf{G}_2 -structures.

\mathfrak{g}	(de^1, \dots, de^7)
37A	$(0, 0, 0, 0, 12, 23, 24)$
37B	$(0, 0, 0, 0, 12, 23, 34)$
37C	$(0, 0, 0, 0, 12 + 34, 23, 24)$
37D	$(0, 0, 0, 0, 12 + 34, 13, 24)$
17	$(0, 0, 0, 0, 0, 12 + 34 + 56)$
37B ₁	$(0, 0, 0, 0, 12 - 34, 13 + 24, 14)$
37D ₁	$(0, 0, 0, 0, 12 - 34, 13 + 24, 14 - 23)$

Table 5: Indecomposable 3-step nilpotent Lie algebras admitting cocalibrated \mathbf{G}_2 -structures.

\mathfrak{g}	(de^1, \dots, de^7)
357A	$(0, 0, 12, 0, 13, 24, 14)$
257A	$(0, 0, 12, 0, 0, 13 + 24, 15)$
257C	$(0, 0, 12, 0, 0, 13 + 24, 25)$
257I	$(0, 0, 12, 0, 0, 13 + 14, 15 + 23)$
257J	$(0, 0, 12, 0, 0, 13 + 24, 15 + 23)$
247A	$(0, 0, 0, 12, 13, 14, 15)$
247B	$(0, 0, 0, 12, 13, 14, 35)$
247C	$(0, 0, 0, 12, 13, 14 + 35, 15)$
247D	$(0, 0, 0, 12, 13, 14, 25 + 34)$
247F	$(0, 0, 0, 12, 13, 24 + 35, 25 + 34)$
247I	$(0, 0, 0, 12, 13, 25 + 34, 35)$
247J	$(0, 0, 0, 12, 13, 15 + 35, 25 + 34)$
247L	$(0, 0, 0, 12, 13, 14 + 23, 15)$
247M	$(0, 0, 0, 12, 13, 14 + 23, 35)$
247N	$(0, 0, 0, 12, 13, 15 + 24, 23)$
247O	$(0, 0, 0, 12, 13, 14 + 35, 15 + 23)$
247P	$(0, 0, 0, 12, 13, 23, 25 + 34)$
247Q	$(0, 0, 0, 12, 13, , 14 + 23, 25 + 34)$
157	$(0, 0, 12, 0, 0, 0, 13 + 24 + 56)$
147A	$(0, 0, 0, 12, 13, 0, 16 + 25 + 34)$
147B	$(0, 0, 0, 12, 13, 0, 14 + 26 + 35)$
147D	$(0, 0, 0, 12, 23, -13, 15 + 26 + 16 - 2 * 34)$
137A	$(0, 0, 0, 0, 12, 34, 15 + 36)$
137B	$(0, 0, 0, 0, 12, 34, 15 + 36 + 24)$
137C	$(0, 0, 0, 0, 12, 14 + 23, 16 - 35)$
257J ₁	$(0, 0, 12, 0, 0, 13 + 14 + 25, 15 + 23)$
247F ₁	$(0, 0, 0, 12, 13, 24 - 35, 25 + 34)$
247P ₁	$(0, 0, 0, 12, 13, 23, 24 + 35)$
147A ₁	$(0, 0, 0, 12, 13, 0, 16 + 24 + 35)$

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137A ₁	(0, 0, 0, 0, 13 + 24, 14 - 23, 15 + 26)
137B ₁	(0, 0, 0, 0, 13 + 24, 14 - 23, 15 + 26 + 24)
147E($\mu \neq 0, 1$)	(0, 0, 0, , 12, 23, -13, $\mu 26 - 15 - (-1 + \mu)34$)
147E ₁ ($\mu > 1$)	(0, 0, 0, 12, 23, -13, $2[26 - 34 - \frac{\mu}{2}16 + \frac{\mu}{2}25]$)

Table 6: Indecomposable 4-step nilpotent Lie algebras admitting cocali-brated \mathbf{G}_2 -structures.

g	(de^1, \dots, de^7)
2457A	(0, 0, 12, 13, 0, 14, 15)
2457B	(0, 0, 12, 13, 0, 25, 14)
2457C	(0, 0, 12, 13, 0, 14 + 25, 15)
2457D	(0, 0, 12, 13, 0, 14 + 23 + 25, 15)
2457E	(0, 0, 12, 13, 0, 23 + 25, 14)
2457F	(0, 0, 12, 13, 0, 14 + 23, 15)
2457G	(0, 0, 12, 13, 0, 15 + 23, 14)
2457H	(0, 0, 12, 13, 0, 23, 14 + 25)
2457I	(0, 0, 12, 13, 0, 14 + 23, 25)
2457J	(0, 0, 12, 13, 0, 14 + 23, 23 + 25)
2457K	(0, 0, 12, 13, 0, 15 + 23, 14 + 25)
2457L	(0, 0, 12, 13, 23, 14 + 25, 15 + 24)
2457M	(0, 0, 12, 13, 23, 24 + 15, 14)
2357A	(0, 0, 0, 12, 14 + 23, 23, 15 - 34)
2357B	(0, 0, 0, 12, 14 + 23, 13, 15 - 34)
2357C	(0, 0, 0, 12, 14 + 23, 24, 15 - 34)
2357D	(0, 0, 12, 14 + 23, 13 + 24, 15 - 34)
1357A	(0, 0, 0, 12, 14 + 23, 0, 15 + 26 - 34)
1357B	(0, 0, 0, 12, 14 + 23, 0, 15 + 36 - 34)
1357C	(0, 0, 0, 12, 14 + 23, 0, 15 + 24 + 36 - 34)
1357D	(0, 0, 12, 0, 23, 24, 16 + 25 + 34)
1357F	(0, 0, 12, 0, 23, 24, 13 + 25 - 46)
1357G	(0, 0, 12, 0, 23, 14, 16 + 25)
1357H	(0, 0, 12, 0, 23, 14, 16 + 25 + 26 - 34)
1357I	(0, 0, 12, 0, 23, 14, 25 + 46)
1357J	(0, 0, 12, 0, 23, 14, 13 + 25 + 46)
1357L	(0, 0, 12, 0, 13 + 24, 23, 16 + 25)
1357O	(0, 0, 12, 0, 13 + 24, 23, 15 + 26 + 34)
1357P	(0, 0, 12, 0, 13 + 24, 23, 15 + 26 + 34)
1357Q	(0, 0, 12, 0, 13, 23 + 24, 15 + 26)
1357R	(0, 0, 12, 0, 13, 23 + 24, 16 + 25 + 34)
2457L ₁	(0, 0, 12, 13, 23, 14 - 25, 15 + 24)
2357D ₁	(0, 0, 0, 12, 14 + 23, 13 - 24, 15 - 34)
1357F ₁	(0, 0, 12, 0, 23, 24, 13 + 25 + 46)
1357P ₁	(0, 0, 12, 0, 13 + 24, 23, 15 + 34 - 26)
1357Q ₁	(0, 0, 12, 0, 13, 23 + 24, 15 - 26)
1357M($\mu \neq 0, -1$)	0, 0, 12, 0, 24 + 13, 14, $-(-1 + \mu)34 + 15 + \mu 26$)
1357N($\mu \neq -2$)	(0, 0, 12, 0, 13 + 24, 14, $46 + 34 + 15 + \mu 23$)
1357S($\mu \neq 1$)	(0, 0, 12, 13, 24 + 23, $25 + 34 + 16 + 15 + \mu 26$)
1357QRS ₁ ($\mu \neq 0$)	(0, 0, 12, 0, 13 + 24, 14 - 23, $\mu 26 + 15 - (-1 + \mu)34$)

Table 7: Indecomposable 5-step nilpotent Lie algebras admitting cocalibrated \mathbf{G}_2 -structures.

\mathfrak{g}	(de^1, \dots, de^7)
23457C	$(0, 0, 12, 13, 14, 15, 25 - 34)$
23457D	$(0, 0, 12, 13, 14, 15 + 23, 25 - 34)$
23457E	$(0, 0, 12, 13, 14 + 23, 15 + 24, 23)$
23457G	$(0, 0, 12, 13, 14 + 23, 15 + 24, 25 - 34)$
13457D	$(0, 0, 12, 13, 14 + 23, 0, 15 + 24 + 26)$
13457F	$(0, 0, 12, 13, 14, 23, 15 + 26)$
12457A	$(0, 0, 12, 13, 0, 14 + 25, 16 + 35)$
12457B	$(0, 0, 12, 13, 0, 14 + 25, 16 + 25 + 35)$
12457C	$(0, 0, 12, 13, 0, 14 + 25, 26 - 34)$
12457D	$(0, 0, 12, 13, 0, 14 + 25, 15 + 26 - 34)$
12457E	$(0, 0, 12, 13, 0, 14 + 23 + 25, 16 + 24 + 35)$
12457F	$(0, 0, 12, 13, 0, 14 + 23 + 25, 26 - 34)$
12457G	$(0, 0, 12, 13, 0, 14 + 23 + 25, 15 + 26 - 34)$
12457H	$(0, 0, 12, 13, 23, 15 + 24, 16 + 34)$
12457I	$(0, 0, 12, 13, 23, 15 + 24, 16 + 34)$
12457J	$(0, 0, 12, 13, 23, 15 + 24, 16 + 14 + 25 + 34)$
12457K	$(0, 0, 12, 13, 23, 15 + 24, 16 + 14 + 34)$
12457L	$(0, 0, 12, 13, 23, 15 + 24, 16 + 26 + 34 - 35)$
12357A	$(0, 0, 0, 12, 14 + 23, 15 - 34, 16 - 35)$
12357B	$(0, 0, 0, 12, 14 + 23, 15 - 34, 16 + 23 - 35)$
12357C	$(0, 0, 0, 12, 14 + 23, 15 - 34, 16 + 24 - 35)$
12457J ₁	$(0, 0, 12, 13, 23, 24 + 15, 16 + 14 - 25 + 34)$
12457L ₁	$(0, 0, 12, 13, 23, -14 - 25, 16 - 35)$
12457N ₁	$(0, 0, 12, 13, 23, -14 - 25, 16 - 35 + 25)$
12357B ₁	$(0, 0, 0, 12, 14 + 23, 15 - 34, 16 - 23 - 35)$
12457N($\mu \in \mathbb{R}$)	$(0, 0, 12, 13, 23, 24 + 15, \mu 25 + 26 + 34 - 35 + 16 + 14)$
123457I($\mu \in \mathbb{R}$)	$(0, 0, 12, 13, 14 + 23, \mu 25 + 26 + 34 - 35 + 16 + 14)$
12457N ₂ ($\mu \geq 0$)	$(0, 0, 12, 13, 23, -14 - 25, 15 + 16 + 24 - 35 + \mu 25)$

Proof. Let \mathfrak{g} be any indecomposable seven dimensional nilpotent real Lie algebra.

If \mathfrak{g} appears in Table 8 then there exists a non zero vector $X \in \mathfrak{z}$ satisfying

$$Z_X^3 \cap \Lambda_-(\mathfrak{h}) = \emptyset,$$

hence, by Corollary 3.2, admits no cocalibrated \mathbf{G}_2 -structures.

Table 8: Indecomposable nilpotent Lie algebras satisfying $Z_X^3 \cap \Lambda_-(\mathfrak{h}) = \emptyset$.

\mathfrak{g}	(de^1, \dots, de^7)	X
2-step nilpotent Lie algebras		
27A	$(0, 0, 0, 0, 0, 12, 14 + 35)$	e_6
27B	$(0, 0, 0, 0, 0, 12 + 34, 15 + 23)$	e_6
3-step nilpotent Lie algebras		
257B	$(0, 0, 12, 0, 0, 13, 14 + 25)$	e_7
257D	$(0, 0, 12, 0, 0, 13 + 24, 14 + 25)$	e_7
257E	$(0, 0, 12, 0, 0, 13 + 45, 24)$	e_7
257G	$(0, 0, 12, 0, 0, 13 + 45, 15 + 24)$	e_7
257H	$(0, 0, 12, 0, 0, 13 + 24, 45)$	e_7
257K	$(0, 0, 12, 0, 0, 13, 23 + 45)$	e_7
257L	$(0, 0, 12, 0, 0, 13 + 24, 23 + 45)$	e_7
247E	$(0, 0, 0, 12, 13, 14 + 15, 25 + 34)$	e_7

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247G	$(0, 0, 0, 12, 13, 14 + 15 + 24 + 35, 25 + 34)$	e_7
247H	$(0, 0, 0, 12, 13, 14 + 24 + 35, 25 + 34)$	e_7
247K	$(0, 0, 0, 12, 13, 14 + 35, 25 + 34)$	e_7
247R	$(0, 0, 0, 12, 13, 14 + 15 + 23, 25 + 34)$	e_7
247E ₁	$(0, 0, 0, 12, 13, 14, 24 + 35)$	e_6
247H ₁	$(0, 0, 0, 12, 13, 14 + 24 - 35, 25 + 34)$	e_7
4-step nilpotent Lie algebras		
1457A	$(0, 0, 12, 13, 0, 0, 14 + 56)$	e_7
1457B	$(0, 0, 12, 13, 0, 0, 23 + 14 + 56)$	e_7
1457E	$(0, 0, 12, 0, 23, 24, 25 + 46)$	e_7
1357M($\mu = -1$)	$(0, 0, 12, 0, 24 + 13, 14, -(1 + \mu)34 + 15 + \mu 26), \mu = -1$	e_7
1357N($\mu = -2$)	$(0, 0, 12, 0, 24 + 13, 14, 46 + 34 + 15 + \mu 23), \mu = -2$	e_7
5-step nilpotent Lie algebras		
23457B	$(0, 0, 12, 13, 14, 25 - 34, 23)$	e_6
23457F	$(0, 0, 12, 13, 14 + 23, 25 - 34, 23)$	e_7
13457A	$(0, 0, 12, 13, 14, 0, 15 + 26)$	e_7
13457B	$(0, 0, 12, 13, 14, 0, 15 + 23 + 26)$	e_7
13457C	$(0, 0, 12, 13, 14, 0, 16 + 25 - 34)$	e_7
13457E	$(0, 0, 12, 13, 14 + 23, 0, 16 + 25 - 34)$	e_7
13457G	$(0, 0, 12, 13, 14, 23, 16 + 24 + 25 - 34)$	e_7
13457I	$(0, 0, 12, 13, 14, 23, 15 + 25 + 26 - 34)$	e_7

If \mathfrak{g} appears in Table 9, as we have seen in the proof of Proposition 4.2 for Lie algebras in Table 3, putting $X = e_7$, $W = \text{Span}(\pi(e_3), \pi(e_4), \pi(e_5), \pi(e_6)) \subset \mathfrak{h}$ and proceeding by contradiction one can prove that it admits no cocalibrated \mathbf{G}_2 -structures.

Table 9: Indecomposable nilpotent Lie algebras satisfying $Z_X^3 \cap \Lambda_-(\mathfrak{h}) \neq \emptyset$ for all $X \in \mathfrak{z}$ but admitting no cocalibrated \mathbf{G}_2 -structures.

\mathfrak{g}
3-step nilpotent Lie algebras
$(0, 0, 12, 0, 13, 23, 14)$
$(0, 0, 12, 0, 13 + 24, 23, 14)$
5-step nilpotent Lie algebra
$(0, 0, 12, 13, 14, 15, 23)$

For any other Lie algebra \mathfrak{g} we can choose a non zero vector $X \in \mathfrak{z}$ and define a triple of forms (ω, ψ_-, η) (see Table 10) which satisfies hypothesis of Proposition 3.1, and consequently defines a cocalibrated \mathbf{G}_2 -structure. Those algebras are listed in tables 4 - 7. \square

6 Concluding remarks

We have thus classified all seven dimensional nilpotent Lie algebras admitting cocalibrated \mathbf{G}_2 -structures. We obtained several such structures according to the fact that, on a closed manifold, if there exists a \mathbf{G}_2 -structure then there exists also a cocalibrated one, [7]. Our classification concerns the invariant case which turns out to be less sporadic, although rich enough to contain many samples.

Now, following [10], we are interested in studying nilsoliton metrics induced by cocalibrated structures, in order to find more restrictive existence conditions. Moreover we hope to analyse the behaviour of certain flows of \mathbf{G}_2 -structures in such context: in particular the modified laplacian co-flow introduced in [15], for which short-time existence is proved.

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7 Appendix

In this appendix we perform some explicit calculations omitted from Propositions 4.2 and 5.1.

Let \mathfrak{g} be a nilpotent seven dimensional real Lie algebra, \mathfrak{z} its center, $0 \neq X \in \mathfrak{z}$ and \mathfrak{h} defined by (3.1).

Fix a basis $\{e_1, \dots, e_7\}$ of \mathfrak{g} and consider an arbitrary closed form $\phi \in \Lambda^4 \mathfrak{g}^*$,

$$\phi = \sum_{i < j < k < l} \phi_{ijkl} e^{ijkl}, \quad \phi_{ijkl} \in \mathbb{R}.$$

$$\mathfrak{g} = (0, 0, 0, 0, 12, 15, 0)$$

Let $X = e_6$, $\mathfrak{h} = (df^1, \dots, df^6) = (0, 0, 0, 0, 12, 0)$, where

$$\begin{aligned} f_i &= \pi(e_i), \quad i < 6, \\ f_6 &= \pi(e_7). \end{aligned}$$

and consider

$$\psi_- = \pi_*(-\iota_{e_6}\phi) = \sum_{i < j < k} \phi_{ij67} f^{ij6} - \phi_{ijk6} f^{ijk}.$$

It turns out that

$$d(\phi) = 0 \iff \begin{cases} \phi_{2346} = 0, \\ \phi_{2367} = 0, \\ \phi_{2467} = 0, \\ \phi_{3456} = 0, \\ \phi_{3457} = 0, \\ \phi_{3467} = 0, \\ \phi_{3567} = 0, \\ \phi_{4567} = 0. \end{cases}$$

Then it results

$$\lambda(\psi_-) = (\phi_{1346}\phi_{2567} + \phi_{1367}\phi_{2467} - \phi_{1467}\phi_{2356})^2 f^{123456} \otimes f^{123456},$$

hence hypothesis of Corollary 3.2 are satisfied. It follows that \mathfrak{g} admits no cocalibrated \mathbf{G}_2 -structures.

$$\mathfrak{g} = (0, 0, 0, 12, 13 - 24, 23 + 14, 0)$$

Let $X = e_7$ and $\mathfrak{h} = (df^1, \dots, df^6) = (0, 0, 0, 12, 13 - 24, 23 + 14)$, where

$$f_i = \pi(e_i), \quad i < 7.$$

It turns out that

$$d(\phi) = 0 \iff \begin{cases} \phi_{1567} = 0, \\ \phi_{2356} = -\phi_{1456}, \\ \phi_{2456} = \phi_{1356}, \\ \phi_{2457} = \phi_{2367} + \phi_{1467} + \phi_{1357}, \\ \phi_{2567} = 0, \\ \phi_{3456} = 0, \\ \phi_{3457} = 0, \\ \phi_{3467} = 0, \\ \phi_{3567} = 0, \\ \phi_{4567} = 0. \end{cases}$$

By way of contradiction suppose ϕ to be stable and consider

$$\begin{aligned} \psi_- &= \pi_*(-\iota_{e_7}\phi) = \sum_{i < j < k} \phi_{ijk7} f^{ijk}, \\ \eta &= \sum_i c_i e^i, \quad c_i \in \mathbb{R}, \end{aligned}$$

where η is the dual form of e_7 with respect the metric induced by ϕ . Then ψ_- lies in $\Lambda_-(\mathfrak{h})$ and for any volume form $\varepsilon \in \Lambda^6 \mathfrak{h}^*$ results

$$J_{\psi_-} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

with respect to the basis $\{f_1, \dots, f_6\}$, hence the 4-dimensional subspace $W = \text{Span}(f_3, \dots, f_6)$ is J_{ψ_-} -invariant.

By assumption the 4-form

$$\sigma = \pi_*(\phi - \pi^*\psi_- \wedge \eta) = \sum_{i < j < k < l} \sigma_{ijkl} f^{ijkl},$$

is the Hodge dual of the fundamental 2-form ω of an $\mathbf{SU}(3)$ -structure on \mathfrak{h} , but it is easy to see that

$$\sigma(f_3, f_4, f_5, f_6) = \sigma_{3456} = \phi_{3456} = 0,$$

or equivalently $\sigma|_W = 0$, contradicting Lemma 3.3. Hence \mathfrak{g} admits no cocalibrated \mathbf{G}_2 -structures.

Table 10: Explicit cocalibrated \mathbf{G}_2 -structures $\phi = \frac{1}{2}\pi^*\omega^2 + (\pi^*\psi_-) \wedge \eta$ on indecomposable nilpotent Lie algebras.

Coefficients C_1, C_2 are chosen so that $d(\frac{1}{2}\omega^2) = \psi_- \wedge d(\eta)$ holds, B so that $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$ holds, and A so that $\det(g) > 0$ holds.

\mathfrak{g}	$\pi^*\omega$	Restrictions
	$\pi^*\psi_-$	
	η	
2-step nilpotent Lie algebras		
37A	$e^{24}-e^{13}+e^{56}$	
	$-e^{125}+e^{236}+e^{146}-e^{345}$	
	e^7	

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37B	$2(e^{13} + e^{24} - e^{67})$	
	$2\sqrt{2}(e^{127} - e^{146} + e^{236} - e^{347})$	
	$e^5 + e^7$	
37C	$e^{12} - e^{34} - e^{67}$	
	$e^{236} - e^{247} - e^{137} - e^{146}$	
	e^5	
37D	$-e^{12} + e^{34} - e^{67}$	
	$-e^{136} - e^{146} + e^{137}$	
	$-e^{147} - e^{246} + e^{237}$	
	e^5	
17	$e^{12} + e^{34} + e^{56}$	
	$-e^{246} + e^{136} + e^{145} + e^{235}$	
	e^7	
37B ₁	$-e^{12} - e^{34} + e^{67}$	
	$-e^{146} + e^{137} - e^{247} - e^{236}$	
	e^5	
37D ₁	$e^{12} - e^{34} - e^{67}$	
	$e^{136} - e^{147} + e^{237} + e^{246}$	
	e_5	
3-step nilpotent Lie algebras		
357A	$e^{25} + e^{14} - e^{36}$	
	$e^{123} - e^{246} + e^{156} - e^{345}$	
	e^7	
257A	$AB(e^{12} + 2e^{34} + e^{56})$	$B = \frac{5A^3}{\sqrt{390}},$ $A > \sqrt{1791}$
	$B^2(\frac{1}{10}e^{125} - \frac{2}{5}e^{134} + 4e^{136} + e^{145}$	
	$+ \frac{1}{5}e^{156} + 4e^{235} - 4e^{246} - \frac{1}{5}e^{345})$	
	$A^2(30e^6 + e^7)$	
257C	$16(e^{13} + e^{25} + e^{46})$	
	$64(e^{124} - e^{156} + e^{236} + e^{345})$	
	$4(-e^3 + e^7)$	
257I	$-e^{12} + e^{35} + e^{45} + e^{46}$	
	$-e^{134} + e^{156} - e^{236} + e^{245} - e^{246}$	
	e^7	
257J	$16(e^{13} + e^{25} + e^{46})$	
	$64(-e^{124} + e^{156} - e^{236} - e^{345})$	
	$4(e^3 + e^7)$	
247A	$e^{12} + e^{34} + e^{56}$	
	$e^{135} - e^{146} - e^{236} - e^{245}$	
	e^7	
247B	$AB(e^{12} + e^{26} - e^{35} + e^{45} + e^{46})$	$B = \frac{8}{25}A^3,$ $0 < A < \sqrt{\frac{7}{40}}$
	$B^2(-e^{123} + e^{124} - 2e^{125} - e^{126} +$	
	$+e^{134} - e^{135} - e^{136} - e^{145} +$	
	$+e^{156} - e^{235} - e^{236} - e^{245})$	
247C	$A^2(3e^5 + e^6 + e^7)$	
	$16(e^{13} + e^{25} + e^{47})$	
	$64(e^{124} - e^{157} + e^{237} + e^{345})$	
	$4(e^4 + e^6)$	
247D	$e^{12} - e^{34} - e^{56}$	
	$e^{136} + e^{145} - e^{235} + e^{246}$	
	e_7	
247F	$\frac{1}{16}(e^{15} + e^{16} - e^{25} - e^{34} + e^{56})$	
	$\frac{1}{64}(e^{123} - e^{124} + e^{145} +$	
	$2e^{235} + e^{236} + e^{246} - e^{356})$	
247I	$\frac{1}{4}(-e^4 + e^7)$	
	$\frac{1}{16}(e^{12} + e^{15} - e^{23} + e^{46})$	
	$64(e^{126} - e^{134} +$	
	$+e^{236} + e^{245} - e^{345})$	
	$\frac{1}{4}(e^4 + e^7)$	

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247J	$e^{13} - e^{25} - e^{46}$ $-e^{124} + e^{126} - e^{145} +$ $+e^{236} + 2e^{345} + e^{356}$ e^7	
247L	$AB(-e^{13} + e^{15} - e^{25} +$ $+2e^{27} - e^{34} + e^{35} +$ $+2e^{45} - 5e^{47})$ $B^2(e^{123} + e^{124} - e^{127} +$ $2e^{134} - 3e^{135} + 8e^{137} + 2e^{147} +$ $-e^{157} - e^{234} - e^{237} - e^{345})$ $A^2(6e^4 + e^6 + 4e^7)$	$B = A^3,$ $0 < A < \frac{1}{\sqrt{452}}$
247M	$AB(e^{15} + e^{23} + e^{35} - 2e^{46})$ $B^2(-2e^{124} - 2e^{126} +$ $-e^{134} - 2e^{136} - e^{145} +$ $+e^{156} + e^{234} + e^{236} + e^{245} + e^{345})$ $A^2(4e^5 + e^7)$	$B = 2A^3,$ $0 < A < \frac{1}{\sqrt{32}}$
247N	$-e^{14} - e^{25} - e^{37}$ $e^{123} - e^{157} + e^{247} - e^{345}$ e^6	
247O	$3\sqrt{3}(-e^{26} + e^{34} + e^{12} + e^{15})$ $3(-e^{123} + e^{146} + e^{236} +$ $+e^{245} - e^{356})$ $e^4 - e^6 + e^7$	
247P	$\frac{1}{16}(e^{17} - e^{23} + e^{25} +$ $-e^{27} + e^{34} + e^{47})$ $\frac{1}{64}(e^{123} + 2e^{134} + 2e^{145} +$ $-e^{147} - e^{245} + e^{357})$ $\frac{1}{4}(e^6 - e^7)$	
247Q	$e^{14} + e^{23} + e^{57}$ $e^{125} - e^{137} + e^{247} + e^{345}$ e^6	
157	$2e^{12} + e^{15} + e^{26} + e^{34} + e^{56}$ $e^{124} + e^{135} - e^{145} +$ $-e^{146} - e^{235} - e^{236} - e^{245}$ e^7	
147A	$AB(\frac{1}{2}e^{12} + e^{34} + \frac{1}{2}e^{56})$ $B^2(-e^{123} + 2e^{136} + e^{145} +$ $+e^{235} - e^{246} + e^{356})$ $A^2(\frac{5}{2}e^4 + e^7)$	$B = \frac{1}{4}A^3$ $A > 5$
147B	$AB(-e^{13} + 4e^{24} + e^{56})$ $B^2(e^{123} + 2e^{124} - 2e^{125} + 4e^{126} +$ $+e^{145} + \frac{1}{2}e^{146} - \frac{1}{2}e^{156} - 2e^{234} +$ $+2e^{235} + 1e^{256} + 1e^{346} - \frac{1}{2}e^{356})$ $A^2(-4e^4 + 6e^5 + e^7)$	$B = \frac{8}{\sqrt{11}}A^3,$ $A > \sqrt{19}$
147D	$AB(-e^{12} - e^{16} - e^{23} +$ $-e^{25} + e^{34} + e^{36} + e^{45})$ $B^2(-e^{123} - 2e^{134} - e^{136} +$ $+e^{145} - e^{246} - e^{356})$ $A^2(e^4 - e^6 + e^7)$	$B = A^3,$ $0 < A < \frac{1}{\sqrt{5}}$
137A	$\frac{2}{3}\sqrt{3}(e^{13} +$ $-e^{24} + e^{56})$ $\frac{4}{3}(e^{126} - e^{145} +$ $-e^{235} - e^{345} + e^{346})$ e^7	
137B	$e^{13} - e^{24} + e^{56}$ $-e^{126} + e^{145} +$ $+e^{235} + e^{345} - e^{346}$ e^7	

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137C	$\frac{2}{3}\sqrt{3}(e^{13}-e^{24}+e^{56})$	
	$\frac{4}{3}(-e^{126}+e^{145}+e^{235}-e^{346})$	
	e^7	
257J ₁	$16(e^{15}+e^{23}+e^{35}-e^{46})$	
	$64(e^{124}+e^{136}-e^{234}-e^{256}-e^{345})$	
	$4(e^3+e^6+e^7)$	
247F ₁	$e^{23}-e^{16}+e^{45}$	
	$-e^{125}+e^{246}+e^{134}-e^{356}$	
	e^7	
247P ₁	$e^{23}+e^{17}+e^{45}$	
	$e^{125}+e^{247}+e^{134}-e^{357}$	
	e^6	
147A ₁	$e^{16}+e^{23}+e^{45}$	
	$e^{124}-e^{135}-e^{256}-e^{346}$	
	e^7	
137A ₁	$\frac{16}{\sqrt{3}}(e^{12}+2e^{34}+2e^{56})$	
	$\frac{16^2}{3}(e^{135}+e^{136}-e^{145}-\frac{3}{2}e^{146}+$ $-e^{236}-\frac{1}{2}e^{245}+e^{246})$	
	e^7	
137B ₁	$AB(e^{12}+e^{34}+e^{56})$	$B=A^3,$ $A>\sqrt{10}$
	$B^2(e^{135}+2e^{136}+e^{145}+$ $+\frac{1}{2}e^{235}-e^{236}-2e^{245}-e^{246})$	
	$A^2(-2e^6+e^7)$	
147E(μ) μ≠0,1	$AB(e^{13}-2e^{26}-e^{34}+e^{45})$	$B=A^3,$ $(A+3+2\mu)(A+3-2\mu)>0$
	$B^2(-e^{123}-2e^{125}-e^{146}+$ $e^{245}+e^{356})$	
	$A^2((3-2\mu)e^6+e^7)$	
147E ₁ (μ) μ>1	$AB(e^{13}+e^{26}+e^{34}-e^{45})$	$B=A^3,$ $A>5$
	$B^2(e^{123}+2e^{125}+e^{146}+e^{245}+e^{356})$	
	$A^2(5e^6+e^7)$	
4-step nilpotent Lie algebras		
2457A	$e^{15}+e^{24}+e^{36}$	
	$e^{123}-e^{146}+e^{256}+e^{345}$	
	e^7	
2457B	$2(-e^{24}+e^{15}-e^{36})$	
	$2\sqrt{2}(e^{123}-e^{146}-e^{256}-e^{345})$	
	$-e^6+e^7$	
2457C	$-e^{26}-e^{46}+e^{23}+e^{15}$	
	$e^{123}+e^{235}+e^{124}+e^{245}+$ $+e^{134}-e^{345}-e^{256}+e^{136}$	
	e^7	
2457D	$16(e^{15}-e^{23}+e^{46})$	
	$64(e^{124}+e^{126}+e^{136}+$ $+e^{245}+e^{256}+e^{345})$	
	$4(e^3+e^7)$	
2457E	$16(e^{15}-e^{23}+e^{47})$	
	$64(e^{124}+e^{137}+e^{257}+e^{345})$	
	$4(-e^4+e^6)$	
2457F	$e^{15}+e^{23}-e^{56}-e^{46}$	
	$e^{125}+e^{124}+e^{136}-e^{256}-e^{345}$	
	e^7	
2457G	$e^{15}-e^{23}+e^{47}$	
	$-e^{124}-e^{137}-e^{257}-e^{345}$	
	e^6	
2457H	$-e^{27}-e^{47}+e^{23}+e^{15}$	
	$-e^{123}+e^{235}+e^{124}+e^{245}+$ $+e^{134}-e^{345}-e^{257}+e^{137}$	
	e^6	

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2457I	$3(e^{15} + e^{23} - e^{56} - e^{46})$	
	$3\sqrt{3}(e^{124} + e^{125} + e^{136} - e^{256} - e^{345})$	
	$e^4 + e^7$	
2457J	$3(e^{15} + e^{23} - e^{56} - e^{46})$	
	$3\sqrt{3}(e^{124} + e^{125} + e^{136} - e^{256} - e^{345})$	
	$e^4 + e^7$	
2457K	$-e^{27} - e^{47} + e^{23} + e^{15}$	
	$-e^{123} + e^{235} + e^{124} + e^{245} + e^{134} - e^{345} - e^{257} + e^{137}$	
	e^6	
2457L	$5(-e^{14} + e^{24} + e^{25} + e^{36})$	
	$5\sqrt{5}(-e^{123} + e^{156} + e^{246} - e^{256} - e^{345})$	
	$-2e^6 + e^7$	
2457M	$e^{15} + e^{24} + e^{37}$	
	$e^{123} - e^{147} + e^{257} + e^{345}$	
	e^6	
2357A	$e^{13} - e^{24} + e^{57}$	
	$-e^{125} - e^{237} - e^{147} - e^{345}$	
	$-\frac{1}{2}e^5 + e^6$	
2357B	$e^{13} - e^{24} + e^{57}$	
	$-e^{125} - e^{237} - e^{147} - e^{345}$	
	e^6	
2357C	$e^{13} - e^{24} + e^{57}$	
	$-e^{125} - e^{237} - e^{147} - e^{345}$	
	e^6	
2357D	$e^{13} - e^{24} + e^{57}$	
	$-e^{125} - e^{237} - e^{147} - e^{345}$	
	e^6	
1357A	$-e^{12} + e^{34} - e^{56}$	
	$e^{135} + e^{136} + e^{145} + e^{146} + e^{235} + e^{246}$	
	e^7	
1357B	$AB(-e^{13} - e^{24} + \frac{1}{4}e^{45} - e^{56})$	$B = A^3,$ $A > \sqrt{3}$
	$B^2(2e^{126} + e^{145} + \frac{1}{2}e^{156} - e^{235} - \frac{1}{2}e^{346})$	
	$A^2(-4e^4 + e^7)$	
1357C	$16(-e^{12} + 2e^{34} - e^{36} + e^{45})$	
	$32(-2e^{124} - 3e^{134} + 4e^{145} + 2e^{156} - 4e^{235} - 2e^{246} - 2e^{346})$	
	$4(e^4 + e^5 + e^7)$	
1357D	$e^{12} - e^{34} - e^{56}$	
	$e^{136} + e^{145} - e^{146} + e^{235} + e^{236} + e^{246}$	
	e^7	
1357F	$256(e^{12} + e^{13} + e^{35} - e^{46} + e^{56})$	
	$4096(e^{123} - e^{124} + e^{125} - 2e^{126} + e^{136} + e^{145} - e^{234} + e^{236} + e^{245} - e^{246} - e^{256})$	
	$16(2e^5 + e^7)$	
1357G	$2(e^{12} - e^{34} - 2e^{56})$	
	$4e^{136} + 4e^{145} - 4e^{146} - 4e^{235} + 4e^{236} - 4e^{245} + 9e^{246}$	
	e^7	
1357H	$2(-e^{12} + 2e^{34} - e^{56})$	
	$4(e^{135} + e^{145} + e^{146} + e^{235} - e^{236} + 2e^{245})$	

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	e^7	
1357I	$3(e^{15} + e^{24} + e^{36})$	
	$3\sqrt{3}(e^{123} - e^{146} + e^{256} + e^{345})$	
	$e^3 + e^6 + e^7$	
1357J	$81(-e^{13} + 2e^{23} + e^{25} - e^{46} + e^{56})$	
	$729(2e^{123} + 3e^{124} - 2e^{125} + e^{136} + e^{146} - e^{256} - e^{345})$	
	$9(2e^3 + 3e^4 - 2e^5 - e^6 + e^7)$	
1357L	$\frac{27}{2}\sqrt{6}(e^{12} + e^{34} + e^{56})$	
	$\frac{81}{2}(6e^{135} - 2e^{136} - 3e^{146} - 3e^{236} - 9e^{245} + 3e^{246})$	
	$9(-e^6 + e^7)$	
1357O	$e^{12} - e^{34} - e^{56}$	
	$e^{136} + \frac{5}{4}e^{145} - \frac{1}{2}e^{146} - e^{235} +$ $-\frac{1}{2}e^{245} + e^{246}$	
	e^7	
1357P	$\frac{1}{\sqrt{2}}(e^{12} + e^{34} + e^{56})$	
	$\frac{1}{2}(-e^{135} + e^{146} - e^{235} +$ $+e^{236} + 2e^{245} + e^{246})$	
	e^7	
1357Q	$e^{12} + e^{34} + e^{56}$	
	$-e^{135} + e^{146} - e^{235} +$ $+e^{236} + e^{245} + e^{246}$	
	e^7	
1357R	$-e^{12} - e^{34} + e^{56}$	
	$-e^{135} - e^{136} + e^{145} +$ $+e^{235} + e^{245} + e^{246}$	
	e^7	
2457L ₁	$e^{15} + e^{24} + e^{36}$	
	$-e^{123} - e^{235} - e^{134} +$ $-2e^{345} - e^{256} + e^{146}$	
	e^7	
2357D ₁	$e^{125} - e^{237} + e^{147} - e^{345}$	
	$e^{13} + e^{24} - e^{57}$	
	e^6	
1357F ₁	$16(e^{12} + e^{35} - e^{46} + e^{56})$	
	$64(-e^{134} + e^{136} + e^{145} - e^{146} +$ $-e^{234} - e^{245} - e^{246} - e^{256})$	
	$4(-e^4 + e^6 + e^7)$	
1357P ₁	$2(e^{12} + 2e^{34} + 2e^{56})$	
	$4(-e^{136} - e^{145} - e^{235} + e^{246})$	
	e^7	
1357Q ₁	$2(-e^{12} - e^{34} + 2e^{56})$	
	$4(-e^{135} + \frac{17}{4}e^{145} - e^{146} + e^{235} +$ $-e^{236} - e^{245} + \frac{9}{4}e^{246})$	
	e^7	
1357M(μ) $\mu(\mu + 1) < 0$	$\frac{(-\mu(\mu+1))^{\frac{3}{2}}}{\mu^2(\mu+1)^2}(-e^{12} - e^{34} + e^{56})$	
	$-\frac{1}{\mu(\mu+1)^4}[(\mu + 1)^3 e^{135} + e^{146} +$ $-e^{235} - \mu e^{236} - (\mu + 1)e^{245} - e^{246}]$	
	e^7	
1357M(μ) $\mu(\mu + 1) > 0$	$AB(e^{12} - e^{34} + e^{36} + e^{45} + \frac{1}{\mu+1}e^{56})$	$B = \frac{(\mu(\mu+1)^{\frac{3}{2}})}{\mu(\mu+1)^3}A^3,$ $A > \frac{\sqrt{2\mu^4+6\mu^3+9\mu^2+6\mu+1}(\mu+1)^3(2\mu+1)}{(\mu^2+2\mu+2)\mu^2}$
	$B^2(e^{135} - e^{146} + (2\mu^2 + 3\mu + 1)e^{234} +$ $+(-\frac{\mu}{\mu+1}e^{235} - \mu e^{236} - (\mu + 1)e^{245} +$ $+ \frac{\mu}{\mu+1}e^{246})$	
	$A^2(\frac{2(\mu+1)^3(\mu+\frac{1}{2})}{\mu(\mu^2+2\mu+2)}e^5 - \frac{2(\mu+1)^2(\mu+\frac{1}{2})}{\mu(\mu^2+2\mu+2)}e^6 + e^7)$	
	$AB(-(\mu + 2)e^{12} - e^{34} + \frac{1}{\mu+2}e^{56})$	

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1357N(μ) $\mu + 2 < 0$	$B^2(-\mu - 2)^{-\frac{3}{2}}(-2e^{126} + 2e^{134} + \frac{4}{\mu+2}e^{135} + 2e^{146} + \frac{2}{\mu+2}e^{156} - \frac{2\mu}{\mu+2}e^{236} +$ $-2e^{245} + \sqrt{(-\mu - 2)}e^{246} + \frac{2}{\mu+2}e^{346})$ $A^2((- \mu^2 - \mu + 2)e^3 + e^7)$	$\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3,$ $A > \frac{-2\mu+2}{\sqrt{-\mu-2}}$
1357N(μ) $\mu + 2 > 0, \mu \neq -1$	$AB(e^{14} + e^{23} + e^{56})$ $B^2(e^{123} - \frac{1}{\mu+1}e^{125} + C_1e^{126} - 2e^{135} +$ $+(\mu + 1)e^{136} - \mu e^{146} - e^{156} + \mu e^{236} + (\mu + 2)e^{245} - e^{346})$ $A^2(C_2e^3 + \frac{1}{\mu+1}e^5 + e^7)$	$d(\frac{1}{2}\omega^2) = \psi_- \wedge d(\eta),$ $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3,$ $\det(g) > 0$
1357N(μ) $\mu = -1$	$AB(e^{12} - 5e^{14} - 2e^{15} - e^{23} +$ $-3e^{34} - e^{35} - e^{46})$ $B^2(2e^{123} - 2e^{124} - 3e^{126} + e^{134} +$ $+2e^{135} + e^{136} + 3e^{146} + e^{156} +$ $+e^{234} + e^{236} - e^{245} + e^{346})$ $A^2(\frac{1}{2}e^2 - e^5 + 3e^6 + e^7)$	$B = A^3,$ $A > \sqrt{\frac{119}{2}}$
1357S(μ) $\mu > 1$	$\mu \sqrt{\frac{(\mu-1)^2}{(\mu+2)^2}}(e^{12} +$ $+\mu e^{34} + e^{56})$ $\mu^2 \frac{(\mu-1)^2}{(\mu+2)^2}(\frac{3}{\mu-1}e^{135} +$ $+e^{136} + e^{145} + e^{235} +$ $-\frac{\mu+2}{\mu-1}e^{236} - e^{245} - e^{246})$ e^7	
1357S(μ) $\mu < 1$	$-\left(\frac{\mu-2}{\mu-1}\right)^{\frac{3}{2}}\frac{(\mu-1)^2}{\mu-2}(-e^{12} +$ $+(-\mu + 2)e^{34} - e^{56})$ $\left(\frac{\mu-2}{\mu-1}\right)^2\left(\frac{(\mu-1)^2}{\mu-2}\right)^2\left(-\frac{1}{\mu-1}e^{135} +$ $-e^{136} + e^{145} - e^{235} +$ $-\frac{\mu-2}{\mu-1}e^{236} + e^{245} - e^{246})$ e^7	
1357QRS ₁ (μ) $\mu \neq 0$	$\sqrt{\frac{\mu^2}{\mu^2+1}}(\frac{1}{\mu}e^{12} + \mu e^{34} + e^{56})$ $\mu^2(\mu^2 + 1)(-\frac{1}{\mu}e^{135} + 2e^{145} + e^{146} +$ $+e^{235} + e^{236} + \frac{1}{\mu}e^{245} - \mu e^{246})$ e^7	
5-step nilpotent Lie algebras		
23457C	$\frac{1}{\sqrt{4}}(e^{16} - 2e^{25} - \frac{1}{4}e^{34} + 4e^{56})$ $\frac{1}{4}(-\frac{3}{4}e^{124} + e^{135} + e^{146} + e^{236} - e^{245})$ e^7	
23457D	$\sqrt{\frac{3}{4}}(-e^{12} - \frac{1}{2}e^{34} + 2e^{56})$ $\frac{3}{4}(e^{135} + e^{146} - e^{235} + e^{236} - e^{245})$ e^7	
23457E	$AB(e^{13} + e^{24} - 2e^{35} +$ $+2e^{37} - e^{45} - e^{57})$ $B^2(\frac{1}{2}e^{124} + 2e^{125} - 2e^{127} + e^{134} +$ $+e^{145} - e^{157} - e^{235} + e^{245} - e^{247})$ $A^2(2e^5 + e^6)$	$B = 2A^3,$ $0 < A < \frac{\sqrt{2}}{4}$
23457G	$AB(e^{15} + e^{24} - e^{34} + e^{36} + e^{45} + 2e^{56})$ $B^2(-3e^{123} - 3e^{125} + e^{135} +$ $+e^{146} + e^{236} - e^{245})$ $A^2(-3e^4 + 2e^6 + e^7)$	$B = A^3,$ $A > \sqrt{11}$
13457D	$\sqrt{\frac{3}{4}}(-e^{12} - \frac{1}{2}e^{34} + 2e^{56})$ $\frac{3}{4}(e^{135} + e^{146} - e^{235} + e^{236} - e^{245})$ e^7	
13457F	$16(e^{12} - e^{35} + e^{46})$ $64(e^{134} + e^{136} + e^{145} +$ $+e^{156} - e^{236} - e^{245})$	

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	$4(-e^4 + e^5 + e^6 + e^7)$	
12457A	$81(e^{15} - e^{23} + e^{24} + e^{26} +$ $-e^{34} - 3e^{36} - e^{46})$	
	$27(-e^{124} + e^{126} - e^{136} + e^{146} +$ $-e^{245} - 2e^{256} - e^{345})$	
	$3(e_1 + e_3 + e_5 - e_6 + e_7)$	
12457B	$AB(e^{15} + e^{23} + 2e^{24} + 2e^{26} +$ $+e^{34} + 2e^{36} - e^{46})$	$B = 3A^3,$ $A > \frac{\sqrt{26}}{3}$
	$B^2(-e^{123} - e^{134} + e^{146} +$ $-2e^{256} - e^{345})$	
	$A^2(-2e^3 + e^6 + e^7)$	
12457C	$AB(-4e^{12} - e^{15} - e^{24} + 4e^{26} +$ $+2e^{36} + e^{46} + e^{56})$	$B = 2A^3,$ $A > \frac{\sqrt{1097}}{4}$
	$B^2(e^{123} + \frac{5}{2}e^{124} + \frac{5}{2}e^{125} +$ $+e^{126} + e^{134} + e^{135} + e^{146} +$ $+e^{234} - e^{235} - \frac{1}{2}e^{256} + \frac{1}{2}e^{345})$	
	$A^2(5e^3 - 4e^4 + 2e^5 + e^7)$	
12457D	$AB(2e^{13} - 3e^{15} - 4e^{26} + e^{35} +$ $+2e^{36} + 2e^{45} + 2e^{46} + e^{56})$	$B = 16A^3,$ $A > \frac{\sqrt{2478}}{2}$
	$B^2(-6e^{124} + 4e^{125} + 3e^{126} +$ $+3e^{134} - e^{135} - 2e^{145} + e^{146} +$ $-2e^{234} - \frac{1}{2}e^{256} + \frac{1}{2}e^{345})$	
	$A^2(-140e^1 + 52e^3 - 14e^4 - 8e^6 + e^7)$	
12457E	$AB(-2e^{12} + e^{15} + 2e^{13} +$ $+e^{24} - e^{25} - e^{46})$	$B = A^3,$ $A > 2\sqrt{5}$
	$B^2(e^{124} + e^{134} + e^{146} - e^{156} +$ $-e^{236} - e^{245} - 2e^{256} - e^{345})$	
	$A^2(\frac{3}{2}e^3 + \frac{1}{2}e^4 + 4e^5 + 2e^6 + e^7)$	
12457F	$AB(2e^{13} - 4e^{15} - e^{24} +$ $-2e^{35} + 2e^{36} - e^{56})$	$B = 6A^3,$ $A > \sqrt{\frac{5}{3}}$
	$B^2(e^{123} + 2e^{126} + 2e^{134} + e^{146})$ $-e^{235} - e^{236} - \frac{1}{2}e^{256} + \frac{1}{2}e^{345})$	
	$A^2(2e^4 + e^7)$	
12457G	$AB(-4e^{13} - e^{24} + 2e^{36} - 2e^{56})$	$B = 8A^3,$ $A > \frac{\sqrt{654}}{8}$
	$B^2(-e^{125} + e^{126} + e^{134} + e^{145} + e^{146} +$ $-e^{235} - e^{236} - \frac{1}{2}e^{256} + \frac{1}{2}e^{345})$	
	$A^2(-6e^3 + 6e^4 + 3e^5 + 2e^6 + e^7)$	
12457H	$-e^{12} - e^{34} + e^{56}$	
	$-2e^{135} + e^{136} + e^{145} - e^{146} +$ $-e^{236} + e^{245}$	
	e^7	
12457I	$\frac{1}{2}(-e^{12} - e^{34} + e^{56})$	
	$\frac{1}{4}(-e^{135} - e^{146} - 2e^{236} + 2e^{245})$	
	e^7	
12457J	$2(-e^{12} + 2e^{34} - e^{56})$	
	$4(e^{135} + e^{146} - e^{236} + e^{245})$	
	$e^5 + e^7$	
12457K	$16(-e^{12} + e^{34} - e^{56})$	
	$64(e^{135} + e^{146} - e^{236} + e^{245})$	
	$4(e^5 + e^7)$	
12457L	$AB(e^{12} + e^{34} + \frac{1}{4}e^{56})$	$B = \frac{2}{\sqrt{139}}A^3,$ $A > \frac{\sqrt{429701402}}{4}$
	$B^2(-\frac{1}{2}e^{125} + e^{134} - e^{135} + 7e^{136} +$ $+7e^{145} + \frac{3}{4}e^{146} - \frac{1}{4}e^{156} - 3e^{234} +$ $+10e^{235} - \frac{1}{4}e^{246} + \frac{3}{4}e^{256} + \frac{1}{2}e^{345})$	
	$A^2(-\frac{4169}{2}e^3 - \frac{247}{4}e^4 +$ $+ \frac{565}{4}e^5 + \frac{27}{2}e^6 + e^7)$	
12357A	$AB(e^{13} + e^{26} - e^{45} + e^{56})$	$B = A^3,$
	$B^2(-e^{124} - 2e^{125} + e^{145} - e^{146} +$	

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	$-2e^{234} - e^{235} - e^{236} - e^{345}$	$A > \sqrt{7}$
	$A^2(3e^4 + e^5 + e^6 + e^7)$	
12357B	$e^{13} - e^{24} + e^{56}$	
	$e^{126} - e^{235} + e^{346} - e^{145}$	
	$-\frac{1}{2}e^5 + e^7$	
12357C	$AB(e^{13} - e^{24} + e^{56})$	$B = A^3,$
	$B^2(-2e^{125} + e^{145} - e^{146} - e^{235} +$	$A > 3$
	$-e^{236} - e^{345})$	
	$A^2(2e^4 + e^6 + e^7)$	
12457J ₁	$AB(e^{13} + e^{24} + e^{34} - e^{56})$	$B = \frac{1}{2}A^3,$
	$B^2(4e^{125} + e^{126} + 2e^{135} + e^{136} +$	$A > 3\sqrt{6}$
	$+e^{145} + e^{146} - 2e^{236} + 2e^{245})$	
	$A^2(\frac{5}{2}e^4 + 2e^5 + e^6 + e^7)$	
12457L ₁	$AB(e^{12} + e^{34} + e^{56})$	$B = \frac{2}{\sqrt{5}}A^3,$
	$B^2(\frac{1}{2}e^{125} + 9e^{135} + 2e^{136} - 2e^{145} +$	$A > 5\sqrt{3}$
	$-\frac{1}{2}e^{146} - e^{234} - 2e^{236} - 2e^{245} +$	
	$+e^{256} - \frac{1}{2}e^{345})$	
	$8e^3 - 4e^4 - 6e^5 - 2e^6 + e^7$	
12457N ₁	$32(-e^{12} - 2e^{24} + e^{26} +$	
	$+2e^{36} - e^{45} - 2e^{56})$	
	$128(2e^{123} - e^{125} + 2e^{135} +$	
	$+e^{146} - e^{234} - 4e^{256} + 2e^{245})$	
	$6e^4 - 2e^5 + 4e^6 + 4e^7$	
12357B ₁	$e^{13} - e^{24} + e^{45} + e^{56}$	
	$-e^{125} - e^{146} + e^{234} - e^{236} - e^{345}$	
	e^7	
12457N(μ) $\mu \in \mathbb{R}$	$AB(e^{15} + e^{23} - e^{36} - e^{46})$	$B = \frac{1}{4}A^3,$
	$B^2(e^{123} - 4e^{124} - e^{125} - 3e^{146} +$	$A > \sqrt{288\mu^2 + 576\mu + 690}$
	$+e^{156} + e^{246} - 3e^{256} - 2e^{345})$	
	$A^2(\frac{3}{2}e^3 + \frac{1}{8}e^4 - \frac{3}{8}e^5 + \frac{3}{2}(-\mu - 1)e^6 + e^7)$	
123457I(μ) $\mu < -1$	$-\frac{1}{\mu+1}(-e^{12} - (\mu+1)e^{34} - e^{56})$	
	$-\frac{1}{(\mu+1)^2}(-e^{135} - e^{146} + e^{236} - e^{245})$	
	e^7	
123457I(μ) $\mu = -1$	$-2e^{12} + 4e^{34} - e^{56}$	
	$4(-2e^{135} - 2e^{146} + e^{236} - e^{245})$	
	e^7	
123457I(μ) $\mu > -1$	$(\mu+1)^2(-e^{12} + (\mu+1)e^{34} - e^{56})$	
	$(\mu+1)^2(e^{135} + e^{146} - e^{236} + e^{245})$	
	e^7	
12457N ₂ (μ) $\mu \geq 0$	$AB(e^{12} - e^{24} + 2e^{36} + e^{45} + 2e^{56})$	$B = 2A^3,$
	$B^2(\frac{1}{2}e^{123} + \frac{1}{2}e^{125} - \frac{3}{2}e^{135} +$	$A > \frac{\sqrt{2\mu^2 + 8\mu + 59}}{4}$
	$+e^{146} - 2e^{256} + e^{345})$	
	$A^2(-\frac{3}{2}e^3 - e^4 - (\mu+2)e^6 + e^7)$	

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